V.1 Paracompactness

Definition 1 Let $X$ be a topological space and $\mathcal{U}$ be a collection of subsets of $X$. $\mathcal{U}$ is **locally finite** if for every $x \in X$ there exists a neighborhood $V_x$ such that $\{U \in \mathcal{U} \mid U \cap V_x \neq \emptyset\}$ is a finite set.

Proposition 1 If $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ is a locally finite collection of open sets, $\{\overline{U}_\alpha \mid \alpha \in J\}$ is also locally finite.

Proof Proof follows from that $U_\alpha \cap V_x = \emptyset \iff \overline{U}_\alpha \cap V_x = \emptyset$. \hfill $\Box$

Proposition 2 If $\{F_\alpha \mid \alpha \in J\}$ is a locally finite collection of closed sets in $X$, $\bigcup_{\alpha \in J} F_\alpha$ is closed in $X$.

Proof We prove that $(\bigcup_{\alpha \in J} F_\alpha)^c$ is open. Suppose $x \notin \bigcup_{\alpha \in J} F_\alpha$ and let $V_x$ be an open neighborhood of $x$ such that $C = \{F_\alpha \mid F_\alpha \cap V_x \neq \emptyset\}$ is a finite collection. Then $F = \bigcup \{F_\alpha \mid F_\alpha \in C\}$ is closed. Thus $W_x = V_x - F$ is an open neighborhood of $x$ and $W_x$ is contained in $(\bigcup_{\alpha \in J} F_\alpha)^c$. Hence $(\bigcup_{\alpha \in J} F_\alpha)^c$ is open. \hfill $\Box$

Proposition 3 If $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ is locally finite, $(\bigcup_{\alpha \in J} U_\alpha)^c$ is open. Suppose $x \notin \bigcup_{\alpha \in J} U_\alpha$ and let $V_x$ be an open neighborhood of $x$ such that $C = \{U_\alpha \mid U_\alpha \cap V_x \neq \emptyset\}$ is a finite collection. Then $F = \bigcup \{U_\alpha \mid U_\alpha \in C\}$ is closed. Thus $W_x = V_x - F$ is an open neighborhood of $x$ and $W_x$ is contained in $(\bigcup_{\alpha \in J} U_\alpha)^c$. Hence $(\bigcup_{\alpha \in J} U_\alpha)^c$ is open.

Definition 2 Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of $X$. $\mathcal{V}$ is a refinement of $\mathcal{U}$ if for all $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$.

Remark When we consider an indexed collection of sets we allow the identical set to be indexed repeatedly. Actually an indexed collection is a family of pairs $(U, \alpha)$.

Definition 3 $X$ is **paracompact** if every open covering of $X$ has a locally finite open refinement that covers $X$.

Example 1. compact $\Rightarrow$ paracompact.

2. $\mathbb{R}^n$ is paracompact:

Let $X = \mathbb{R}^n$ and $\mathcal{A}$ be an open covering of $X$. Let $B_0 = \emptyset$ and $B_m$ be an open ball of radius $m$ centered at the origin for each $m = 1, 2, \ldots$. For $B_m$ choose $A_1, \ldots, A_{k_m} \in \mathcal{A}$.
which covers $B_m$. Let 

$$A'_i = A_i \cap (X - B_{m-2})$$

and 

$$\mathcal{C}_m = \{A'_1, \ldots, A'_{k_m}\}.$$ 

Now $\mathcal{C} = \bigcup \mathcal{C}_m$ is a refinement of $\mathcal{A}$ and a locally finite open covering of $X$.

**Proposition 4** Suppose that $X$ is a paracompact space. For all open covering $U = \{U_\alpha \mid \alpha \in J\}$ of $X$, there exists a locally finite precise open refinement $W = \{W_\alpha \mid \alpha \in J\}$ of $U$ that covers $X$. (“precise” means that $U$ and $W$ have the same index set $J$ and $W_\alpha \subset U_\alpha, \forall \alpha \in J$.)

**Proof** Since $X$ is paracompact, there exists a locally finite open refinement $V = \{V_\beta \mid \beta \in K\}$ of $U$. For each $\beta \in K$, there exists $\alpha \in J$ such that $V_\beta$ is contained in $U_\alpha$. Define 

$$\varphi : K \to J \text{ as } \alpha = \varphi(\beta).$$

Also let 

$$W_\alpha := \bigcup \{V_\beta \mid \alpha = \varphi(\beta)\}.$$ 

Then $W_\alpha \subset U_\alpha$ and $W := \{W_\alpha \mid \alpha \in J\}$ is locally finite since $V$ is locally finite. $\square$

**Proposition 5** If a topological space $X$ is paracompact Hausdorff, $X$ is normal.

**Proof**

**Step 1 : $X$ is regular.** Let $A$ be a closed subset of $X$ and $x$ be any point which is not in $A$. Since $X$ is a Hausdorff space, $x \in A^c$ and $a \in A$ can be separated by two disjoint open sets $U_a$ and $U_x$ so that $x \notin U_a$. Define 

$$\mathcal{U} = \{U_a \mid a \in A\}.$$ 

Then $\mathcal{U}$ is an open covering of $A$. Thus 

$$\mathcal{U} \cup \{A^c\}$$

is an open covering of $X$. There exists a precise locally finite open refinement $\mathcal{V} \cup \{G\}$ of $\mathcal{U} \cup \{A^c\}$ that covers $X$, where 

$$\mathcal{V} = \{V_a \mid a \in A\} \text{ and } G \subset A^c.$$
Let $V$ be the union of all the elements in $\mathcal{V}$. Now $A$ is contained in $V$ and thus in $\overline{V}$. Recall that $\mathcal{V} = \bigcup \{ V_a \mid a \in A \}$ since $\mathcal{V}$ is locally finite. Since each $V_a$ is in $\overline{U}_a$,

$$A \subset V \subset \mathcal{V} \subset \bigcup U_a.$$  

So $\overline{V}$ is disjoint from $x$, namely we can separate $x$ and $A$ using $V$ and $\overline{V}$.

*Step 2 : $X$ is normal.* Suppose $A$ and $B$ are two disjoint closed sets in $X$. Since $X$ is regular, a point $a$ of $A$ and $B$ can be separated by two open sets. Paracompactness of $X$ enables us to construct a locally finite open covering of $A$ which is disjoint from $B$. Repeat exactly the same procedure in *Step 1* to obtain two disjoint open neighborhoods of $A$ and $B$.

**Proposition 6 (Shrinking lemma)** Suppose $X$ is paracompact Hausdorff; Then for any collection $\mathcal{U} = \{ U_\alpha \mid \alpha \in J \}$ of open subsets of $X$ which covers $X$, there exists a locally finite precise open refinement $\mathcal{V} = \{ V_\alpha \mid \alpha \in J \}$ which covers $X$ such that $V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$ for each $\alpha \in J$.

**Proof**

For each $x$ there exists $U_\alpha$ containing $x$ and an open neighborhood $O_x$ of $x$ such that

$$x \in O_x \subset \overline{O}_x \subset U_\alpha.$$  

Let

$$\varphi : X \to J \quad \text{as} \quad \alpha = \varphi(x).$$  

Using Proposition 4, we can construct a precise locally finite open refinement $\mathcal{W} = \{ W_x \mid x \in X \}$ of $\{ O_x \mid x \in X \}$ which covers $X$. Let

$$V_\alpha = \bigcup \{ W_x \mid \varphi(x) = \alpha \}$$  

for each $\alpha \in J$. Note that

$$W_x \subset O_x \subset \overline{O}_x \subset U_\alpha.$$  

Thus $V_\alpha \subset U_\alpha$. Now $\mathcal{V} = \{ V_\alpha \mid \alpha \in J \}$ is a locally finite precise open refinement of $\mathcal{U}$ which covers $X$ and

$$\overline{V}_\alpha = \bigcup_{\alpha = \varphi(x)} W_x \subset \bigcup_{\alpha = \varphi(x)} \overline{O}_x \subset U_\alpha.$$  

$\square$
Definition 4 Let $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ be an open covering of $X$. An indexed family of continuous functions

$$ \phi_\alpha : X \to [0, 1] $$

is said to be a **partition of unity** on $X$ subordinate to $\{U_\alpha\}$ if

1. $\text{support}\phi_\alpha$ is contained in $U_\alpha$.
2. $\{\text{support}\phi_\alpha \mid \alpha \in J\}$ is locally finite.
3. $\Sigma_\alpha \phi_\alpha(x) = 1$ for each $x$.

**Remark**  support $f$ is the closure of $\{x \in X \mid f(x) \neq 0\}$

Theorem 7 (Existence of partition of unity) If $X$ is a paracompact Hausdorff space, any open covering $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ has a partition of unity $\{f_\alpha\}$ subordinate to $\mathcal{U}$.

**Proof** Shrink $\mathcal{U}$ to get a precise locally finite open refinement $\mathcal{V} = \{V_\alpha\}$ that covers $X$. Shrink $\mathcal{V}$ once more to get $\mathcal{W} = \{W_\alpha\}$ using the shrinking lemma. Thus $W_\alpha \subset V_\alpha \subset \overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in J$.

By Urysohn’s lemma, there exists $g_\alpha : X \to [0, 1]$ such that $g_\alpha(\overline{W_\alpha}) = \{1\}$ and $g_\alpha(V_\alpha^c) = \{0\}$. If $W_\alpha = \emptyset$, $g_\alpha \equiv 0$. Since $\text{support} \ g_\alpha \subset \overline{V_\alpha} \subset U_\alpha$, $\{\text{support} \ g_\alpha \mid \alpha \in J\}$ is locally finite. Thus $\Sigma_\alpha g_\alpha$ is a well-defined continuous function such that $\Sigma_\alpha g_\alpha \geq 1$ since $\mathcal{W} = \{W_\alpha\}$ is a covering and $g_\alpha(\overline{W_\alpha}) = 1$.

Define

$$ f_\alpha := \frac{g_\alpha}{\Sigma g_\alpha} $$

then

$$ \text{support} \ f_\alpha = \text{support} \ g_\alpha \text{ and } \Sigma f_\alpha \equiv 1. $$

**Remark** 1. A product of paracompact spaces need not to be paracompact. ($\mathbb{R}^J$)

Also a subspace of paracompact space need not to be paracompact, but a closed subspace is paracompact obviously.

2. See Munkres for Stone’s theorem and Smirnov metrization theorem.

**Homework** Show that the product of paracompact space and a compact space is paracompact.