II.1 Separation Axioms

정의 1 A topological space $X$ is called a Hausdorff space ($T_2$ - space) if each two disjoint points have non-intersecting neighborhoods, i.e., for each $x, y$, there exist $O_x, O_y$ which are open sets with $x \in O_x$ and $y \in O_y$ such that $O_x \cap O_y = \emptyset$.

정의 2 A topological space $X$ is said to be $T_1$, if for each pair of distinct point, each has a neighborhood which does not contain the other.

A space $X$ is said to be regular, if for each pair consisting of a point $x$ and a closed set $B$ disjoint from $x$, there exist disjoint open sets containing $x$ and $B$, respectively. ($T_3$)

A space $X$ is said to be normal, if for each pair $A, B$ of disjoint closed sets of $X$, there exist disjoint open sets containing $A$ and $B$, respectively. ($T_4$)

Example A discrete space is Hausdorff.
A metric space is Hausdorff.
A indiscrete space is not Hausdorff.
A space with cofinite topology is not Hausdorff but is $T_1$.

명제 1 (1) Each subspace of a Hausdorff space is Hausdorff.
(2) $\prod X_\alpha$ is Hausdorff if and only if each $X_\alpha$ is Hausdorff.

증명 (1) Let $X$ be a Hausdorff space and $Y$ be a subspace of $X$. Let $a, b \in Y \subset X$ with $a \neq b$. Since $X$ is Hausdorff, there are disjoint open neighborhoods $U$ and $V$, containing $a$ and $b$, respectively. By definition of subspace, $Y \cap U$ and $Y \cap V$ are disjoint open neighborhoods in $Y$ containing $a$ and $b$, respectively.

(2) ($\Leftarrow$) Let $X = \prod X_\alpha$. Let $x = (x_\alpha)$, $y = (y_\alpha)$ with $x_\alpha \neq y_\alpha$ for some $\alpha$. Since $X_\alpha$ is Hausdorff, there are separating open neighborhoods $O_{x_\alpha}$ and $O_{y_\alpha}$. Then $p^{-1}(O_{x_\alpha})$ and $p^{-1}(O_{y_\alpha})$ are separating open neighborhoods in $X$.

($\Rightarrow$) Since $X_\alpha$ can be embedded as a subspace of $\prod X_\alpha$ which is Hausdorff, $X_\alpha$
is also Hausdorff by (1).

**exercise** For each $\beta \neq \alpha$, fix a point $a_\beta \in X_\beta$. Then $s : X_\alpha \rightarrow \prod X_\alpha$ given by

$$s(x_\alpha)_\beta = \begin{cases} a_\beta & \beta \neq \alpha \\ x_\alpha & \beta = \alpha \end{cases}$$

is an embedding.

**Proof**

**명제 2** $X$ is a Hausdorff space if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

**증명** $X$ is Hausdorff.

$\iff \forall (x, y) \in \Delta^c, \exists$ Open neighborhoods $U_x, U_y$ of $x$ and $y$ s.t. $U_x \times U_y \subset \Delta^c$.

$\iff \Delta^c$ is open in $X \times X$.

$\iff \Delta$ is closed in $X \times X$.  

**명제 3** Suppose that $X$ is Hausdorff, then the followings hold.

1. Each point in $X$ is closed
2. If $x$ is an accumulation point of $A$ in $X$, then each neighborhood of $x$ contains infinitely many points of $A$

**증명**

(1) Clear by definition.

(2) Suppose $U$ is an open set containing $x$ and only finite number of points of $A$ different from $x$. Since $B := U \cap A - \{x\}$ is a finite subset of a Hausdorff space, it is closed and hence $V := U - B$ is open. Then $V$ is a neighborhoods of $x$ containing no points of $A$ different from $x$. Thus $x$ is not an accumulation point, which is a contradiction.

**명제 4** Let $f, g : X \rightarrow Y$ be continuous maps from a topological space $X$ to a Hausdorff space $Y$. Then

1. $\{x \mid f(x) = g(x)\}$ is closed
2. If $D \subset X$ is dense, i.e., $\overline{D} = X$ and $f |_D = g |_D$, then $f = g$ on $X$
3. The graph of $f$ is closed in $X \times Y$
증명 (1) Define $\varphi : X \rightarrow Y \times Y$ by $\varphi : x \mapsto (f(x), g(x))$, then $\{x \mid f(x) = g(x)\} = \varphi^{-1}(\Delta)$. Since $Y$ is Hausdorff, Thus $\Delta$ is closed. Since $\varphi$ is continuous, $\varphi^{-1}(\Delta)$ is closed.

(2) Since $f \mid_D = g \mid_D$, $D \subset \{x : f(x) = g(x)\}$. Since $\{x : f(x) = g(x)\}$ is closed, $X = D \subset \{x : f(x) = g(x)\} \subset X$. Thus $f = g$ on $X$.

(3) Define $\psi : X \times Y \rightarrow Y \times Y$ by $\psi : (x, y) \mapsto (f(x), y)$. Then the graph of $f = \{(x, y) : f(x) = y\}$ is equal to $\psi^{-1}(\Delta)$. Thus the graph of $f$ is closed.

Homework 1 Suppose $Y$ is not Hausdorff in the preceding proposition. Find counter examples to (1) and (2) above.